Two equivalent definitions of integration $\int_X f\,\mathrm{d}\mu$, using $U(f,\mu) = L(f,\mu)$, upper sum equals lower sum

In the class, following the line of defining $\int_X f\,\mathrm{d}\mu$ using $U(f,\mu) = L(f,\mu)$, we can give the following two defintions.

Definition A. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f\colon X \to \mathbb{R}$ be a function. We say that the integration of f exists if $U(f,\mu) = L(f,\mu)$, where

$$U(f,\mu) = \inf\left\{\int_X s\,\mathrm{d}\mu\colon s \text{ is simple and } s \geq f\right\}$$

and

$$L(f,\mu) = \sup\left\{\int_X s\,\mathrm{d}\mu\colon s \text{ is simple and } s \leq f\right\}.$$

Note that the s above can take value $+\infty$ and $-\infty$. In this case, we use A-$\int_X f\,\mathrm{d}\mu$ to denote the integration of f, whose value is just the value of $U(f,\mu) = L(f,\mu)$.

Definition B. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f\colon X \to \mathbb{R}$ be a function. Use f_+ and f_- to denote the positive part and negative part of f, as defined in class. We say that the integration of f exists is if both A-$\int_X f_+\,\mathrm{d}\mu$ and A-$\int_X f_-\,\mathrm{d}\mu$ exist and at most one of them if ∞. In this case, the integration of f, denoted as B-$\int_X f\,\mathrm{d}\mu$, is defined to be

$$\text{B-}\int_X f\,\mathrm{d}\mu = \text{B-}\int_X f_+\,\mathrm{d}\mu - \text{B-}\int_X f_-\,\mathrm{d}\mu.$$

As I mentioned in the class, for measurable functions, these two defintions are just equivalent if the integration if finite.

Remark: One key step we need is: starting from $U(f,\mu) = L(f,\mu)$, we can, for any $\epsilon > 0$, find a pair of simple functions g and h, such that $g \leq f \leq h$ and $\int_X h\,\mathrm{d}\mu - \int_X g\,\mathrm{d}\mu < \epsilon$. To ensure this, we assume f to have finite integration, that is, we have things like $U(f,\mu) = L(f,\mu) \in (-\infty, +\infty)$.

Now, we will show the equivalence of them.

Proposition 1. For any measurable positive function f, the integration of f exists in the sense of

Defn. A if and only if the integration of f exists in the sense of Defn. B. Besides, their integrations in two definitions are the same, if exists.

This proposition 1 is just trivial.

Before stating the next proposition, we will mention the following fact.

Fact 2. For any measurable function f, the following are equivalent:

1) $U(f, \mu) = L(f, \mu) < \infty$

2) $U(f_+, \mu) = L(f_+, \mu) < \infty$ and $U(f_-, \mu) = L(f_-, \mu) < \infty$.

A sketch of the proof is like this: Let E be measurable subset of X on which f is positive. Any simple function restricted on E or $X - E$ is still a simple function. For the other direction, if one simple function is on E and another simple function is on $X - D$ (both satisfying $\geq f$ or $\leq f$), combining them together, you can get a simple function on X satisfying $\geq f$ or $\leq f$. Using the definition of integrations, you should be able to finish the proof.

From this fact, it follows directly that the integration of f in Defn. A is finite if and only if the integration of f in Defn. B is finite. So we can just say "the integration of f is finite" without stating in which definition.

Proposition 3. In case of finite integrations, the above two definitions (Defn. A and Defn. B) are equivalent.

Proof:

"only if" part:

Assume the finite integration of a measurable function f exists in the sense of Defn. A. We will first show that the integration of f exists in the sense of Defn. B. Then we will show that

$$\text{A-}\int_X f \, \mathrm{d}\mu = \text{B-}\int_X f \, \mathrm{d}\mu$$

Let $X^+ = \{x \colon f(x) \geq 0\}$ and let $X^- = \{x \colon f(x) < 0\}$. Then both X^+ and X^- are measurable, $X^+ \cap X^- = 0$ and $X^+ \cup X^- = X$. Easy to check that $\chi_{X^+} \cdot f_+ = f_+$, $\chi_{X^-} \cdot f_- = f_-$, $\chi_{X^+} \cdot f = f_+$ and $\chi_{X^-} \cdot f = -f_-$.

According to Defn. A, we know that $U(f, \mu) = L(f, \mu)$. As $\chi_{X^+} \cdot f = f_+$, $\chi_{X^-} \cdot f = -f_-$ and note that $U(-f_-, \mu) = -L(f_-, \mu)$ and $L(-f_-, \mu) = -U(f_-, \mu)$, in order to show the integration of f exists in the sense of Defn. B, we just need to show the integrations of $\chi_{X^+} \cdot f$ and $\chi_{X^-} \cdot f$ both exist (as these two funtions are either postiive or negative, as for their integrations, Defn. A and Defn. B are just the same). *If you are still concerned about why Defn. A and Defn. B agrees for negative functions, recall that we have proved something like $U(-F, \mu) = -L(F, \mu)$ and $L(-F, \mu) = -U(F, \mu)$ for any measurable function F couple of weeks ago in class. This enables us to transform the case of negative functions to positive functions.*

We will just show the that the integration of $\chi_{X^+} \cdot f$ exists in the sense of Defn. B, as the case of $\chi_{X^-} \cdot f$ is just similar.

As the integration of f exists in the sense of Defn. A and the integration is finite, for any $\epsilon > 0$, there exists simple functions g and h, such that $g \leq f \leq h$, and $\int_X h \, \mathrm{d}\mu - \int_X g \, \mathrm{d}\mu < \epsilon$. As $\chi_{X^+} \geq 0$, we have

$$\chi_{X^+} \cdot g \leq \chi_{X^+} f \leq \chi_{X^+} \cdot h.$$

That is

$$\chi_{X^+} \cdot g \leq f_+ \leq \chi_{X^+} \cdot h,$$

where $\chi_{X^+} \cdot g$ and $\chi_{X^+} \cdot h$ are simple functions satisfying (as we are dealing with simple functions)

$$\begin{aligned}
\int_X \chi_{X^+} \cdot h \, \mathrm{d}\mu - \int_X \chi_{X^+} \cdot g \, \mathrm{d}\mu &= \int_X \chi_{X^+} \cdot h - \chi_{X^+} \cdot g \, \mathrm{d}\mu \\
&= \int_X \chi_{X^+} \cdot (h - g) \, \mathrm{d}\mu \\
&= \int_{X^+} h - g \, \mathrm{d}\mu \\
&\leq \int_X h - g \, \mathrm{d}\mu \\
&= \epsilon.
\end{aligned}$$

It then follows that the integration of f_+ exists. Similarly, the integration of f_- exists. Thus the integration of f exists in the sense of Defn. B.

Now, it remains to show that A-$\int_X f \, \mathrm{d}\mu =$ B-$\int_X f \, \mathrm{d}\mu$. This follows directly from the sketchy proof of the above listed Fact 2 (why?).

"if" part:

Assume the finite integration of f exists in the sense of Defn. B, we will show that the integration of f exists in the sense of Defn. A, and A- $\int_X f \,\mathrm{d}\mu =$ B- $\int_X f \,\mathrm{d}\mu$.

According to Defn. B, the integraion of f_+ and f_- both exists in the sense of Defn. A. (for simple or negative functions, Defn. A and Defn. B are just the same). As the integration is finite, for any $\epsilon > 0$, there exists simple functions g_1 and h_1, such that

$$g_1 \leq f_+ \leq h_1 \quad \text{and} \quad \int_X h_1 \,\mathrm{d}\mu - \int_X g_1 \,\mathrm{d}\mu < \epsilon.$$

We can furthur assume that g_1 and h_1 are all zero on $\{x\colon f(x) < 0\}$. In fact, if not, just multiplying g_1 and h_1 with $\chi_{\{x\colon f(x) \geq 0\}}$ will do the trick.

Similarly, we can find two simple functions g_2, h_2, such that

$$g_2 \leq f_- \leq h_2 \quad \text{and} \quad \int_X h_2 \,\mathrm{d}\mu - \int_X g_2 \,\mathrm{d}\mu < \epsilon$$

and g_2 and h_2 are all zero on $\{x\colon f(x) \geq 0\}$.

Easy to check that $g_1 - h_2$ and $h_1 - g_2$ are simple funcitons, satisfying

$$g_1 - h_2 \leq f = f_+ - f_- \leq h_1 \leq h_1 - g_2 \quad \text{and} \quad \int_X h_1 - g_2 \,\mathrm{d}\mu - \int_X g_1 - h_2 \,\mathrm{d}\mu < 2\epsilon.$$

Let $\epsilon \to 0$, we are done. That is, we proved both the existence of the integration of f in the sense of Defn. A and the desired fact that A- $\int_X f \,\mathrm{d}\mu =$ B- $\int_X f \,\mathrm{d}\mu$.

Q.E.D.